

On Bregman Voronoi Diagrams

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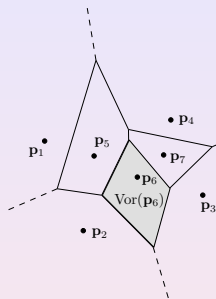
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Ordinary Voronoi Diagrams



Voronoi diagram $\text{Vor}(\mathcal{S})$ s.t.

$$\text{Vor}(\mathbf{p}_i) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{p}_i \mathbf{x}\| \leq \|\mathbf{p}_j \mathbf{x}\| \forall \mathbf{p}_j \in \mathcal{S}\}$$

- Voronoi sites (static view).
- Voronoi generators (dynamic view).



→ René Descartes, 17th century.

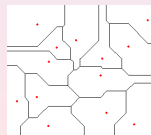
→ Partition the Euclidean space \mathbb{E}^d wrt

the Euclidean distance $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^d x_i^2}$.

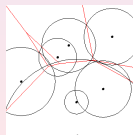
Generalizing Voronoi Diagrams

Voronoi diagrams *widely studied* in comp. geometry [AK'00]:

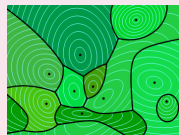
- Manhattan (taxi-cab) diagram (L_1 norm): $\|\mathbf{x}\|_1 = \sum_{i=1}^d |x_i|$,
- Affine diagram (power distance): $\|\mathbf{x} - \mathbf{c}_i\|^2 - r_i^2$,
- *Anisotropic diagram* (quad. dist.): $\sqrt{(\mathbf{x} - \mathbf{c}_i)^T \mathbf{Q}_i (\mathbf{x} - \mathbf{c}_i)}$,
- *Apollonius diagram* (circle distance): $\|\mathbf{x} - \mathbf{c}_i\| - r_i$,
- *Möbius diagram* (weighted distance): $\lambda_i \|\mathbf{x} - \mathbf{c}_i\| - \mu_i$,
- Abstract Voronoi diagrams [Klein'89], etc.



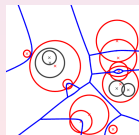
Taxi-cab diagram



Power diagram



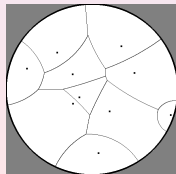
Anisotropic diagram



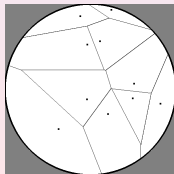
Apollonius diagram

Non-Euclidean Voronoi diagrams

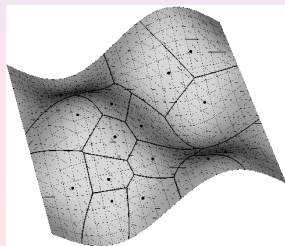
- Hyperbolic Voronoi: Poincaré disk [B+'96], Poincaré half-plane [OT'96], etc.
- Statistical Voronoi diagrams (Kullback-Leibler divergence) [OI'96] & [S+'98]
Divergence between two statistical distributions
$$\text{KL}(p||q) = \int_x p(x) \log \frac{p(x)}{q(x)} dx$$
 (relative entropy)
- Riemannian Voronoi diagrams: geodesic length (aka geodesic Voronoi diagrams) [LL'00]



Hyperbolic Voronoi (Poincaré)



Hyperbolic Voronoi (Klein)



Riemannian Voronoi

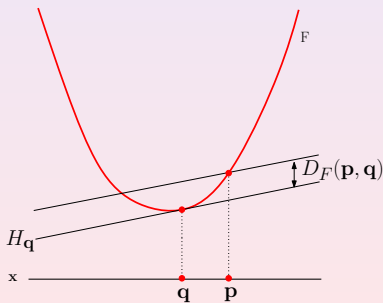
Bregman divergences

F a strictly convex and differentiable function defined over a convex set domain \mathcal{X}

$$D_F(\mathbf{p}, \mathbf{q}) = F(\mathbf{p}) - F(\mathbf{q}) - \langle \mathbf{p} - \mathbf{q}, \nabla F(\mathbf{q}) \rangle$$

not a distance

(not necessarily symmetric nor does triangle inequality hold)



Example: The squared Euclidean distance

- $F(x) = x^2$: strictly convex and differentiable over \mathbb{R}^d

(Multivariate $F(\mathbf{x}) = \sum_{i=1}^d x_i^2$)

$$\begin{aligned}D_F(\mathbf{p}, \mathbf{q}) &= F(\mathbf{p}) - F(\mathbf{q}) - \langle \mathbf{p} - \mathbf{q}, \nabla F(\mathbf{q}) \rangle \\ &= \mathbf{p}^2 - \mathbf{q}^2 - \langle \mathbf{p} - \mathbf{q}, 2\mathbf{q} \rangle \\ &= \|\mathbf{p} - \mathbf{q}\|^2\end{aligned}$$

Voronoi diagram equivalence classes

Since $\text{Vor}(\mathcal{S}; d_2) = \text{Vor}(\mathcal{S}; d_2^2)$, the ordinary Voronoi diagram is interpreted as a Bregman Voronoi diagram.

(Any strictly monotone function f of d_2 yields the same ordinary Voronoi diagram: $\text{Vor}(\mathcal{S}; d_2) = \text{Vor}(\mathcal{S}; f(d_2))$.)

Bregman divergences for probability distributions

- $F(\mathbf{p}) = \int p(x) \log p(x) dx$ (Shannon entropy)

(Discrete distributions $F(\mathbf{p}) = \sum_x p(x) \log p(x)$)

$$\begin{aligned} D_F(\mathbf{p}, \mathbf{q}) &= \int (p(x) \log p(x) - q(x) \log q(x) \\ &\quad - \langle p(x) - q(x), \log q(x) + 1 \rangle) dx \\ &= \int p(x) \log \frac{p(x)}{q(x)} dx \quad (\text{KL divergence}) \end{aligned}$$

Kullback-Leiber divergence also known as:
relative entropy or I -divergence.

(Defined either on the probability simplex or *extended* on the full positive quadrant.)

Bregman divergences for sound processing

$$F(\mathbf{p}) = - \int_x \log p(x) dx$$

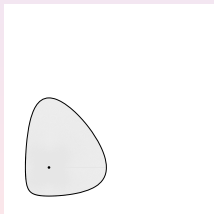
(Burg entropy)

$$D_F(\mathbf{p}, \mathbf{q}) = \int_x \left(\frac{p(x)}{q(x)} - \log \frac{p(x)}{q(x)} - 1 \right) dx$$

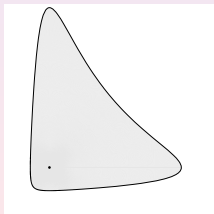
(Itakura-Saito)

Convexity & Bregman balls

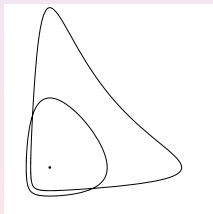
$D_F(\mathbf{p}||\mathbf{q})$ is *convex* in its first argument \mathbf{p} but *not necessarily* in its second argument \mathbf{q} .



$$\text{ball}(\mathbf{c}, r) = \{\mathbf{x} \mid D_F(\mathbf{x}, \mathbf{c}) \leq r\}$$



$$\text{ball}'(\mathbf{c}, r) = \{\mathbf{x} \mid D_F(\mathbf{c}, \mathbf{x}) \leq r\}$$



Superposition of I.-S. balls

Dual divergence

Convex conjugate of F

Unique *convex conjugate* function $G (= F^*)$ obtained by the Legendre transformation: $G(\mathbf{y}) = \sup_{\mathbf{x} \in \mathcal{X}} \{\langle \mathbf{y}, \mathbf{x} \rangle - F(\mathbf{x})\}$.

$$\nabla G(\mathbf{y}) = \nabla(\langle \mathbf{y}, \mathbf{x} \rangle - F(\mathbf{x})) = 0 \rightarrow \boxed{\mathbf{y} = \nabla F(\mathbf{x})}.$$

(thus we have $\mathbf{x} = \nabla F^{-1}(\mathbf{y})$)

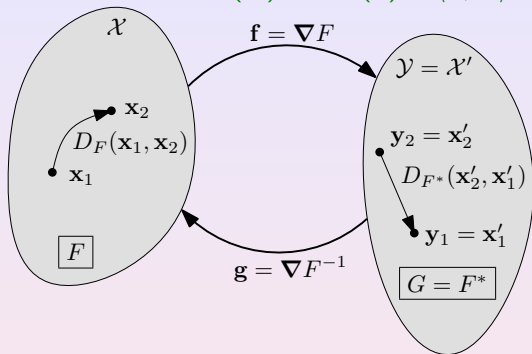
$D_F(\mathbf{p}||\mathbf{q}) = F(\mathbf{p}) - F(\mathbf{q}) - \langle \mathbf{p} - \mathbf{q}, \mathbf{q}' \rangle$ with $(\mathbf{q}' = \nabla F(\mathbf{q}))$. F^* ($= G$) is a Bregman generator function such that $(F^*)^* = F$.

Dual Bregman divergence

$$D_F(\mathbf{p}||\mathbf{q}) = F(\mathbf{p}) + F^*(\mathbf{q}') - \langle \mathbf{p}, \mathbf{q}' \rangle = D_{F^*}(\mathbf{q}'||\mathbf{p}')$$

Convex conjugate and Dual Bregman divergence

Legendre transformation: $F^*(\mathbf{x}') = -F(\mathbf{x}) + \langle \mathbf{x}, \mathbf{x}' \rangle$.



$$\begin{aligned} D_F(\mathbf{x}_1, \mathbf{x}_2) &= F(\mathbf{x}_1) - F(\mathbf{x}_2) - \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}'_2 \rangle \\ &= -F^*(\mathbf{x}'_1) + \langle \mathbf{x}_1, \mathbf{x}'_1 \rangle + F^*(\mathbf{x}'_2) - \langle \mathbf{x}_1, \mathbf{x}'_2 \rangle \\ &= D_{F^*}(\mathbf{x}'_2, \mathbf{x}'_1) \end{aligned}$$

Exponential families in Statistics

Canonical representation of the proba. density func. of a r.v. X

$$p(x|\theta) \stackrel{\text{def}}{=} \exp\{\langle \theta, \mathbf{f}(x) \rangle - F(\theta) - k(\mathbf{f}(x))\},$$

with $\mathbf{f}(x)$: sufficient statistics and θ : natural parameters.

F : cumulant function (or log-partition function).

Example: Univariate Gaussian distributions $\mathcal{N}(\mu, \sigma)$

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} = \exp\left\{\langle [x \ x^2]^T, [\frac{\mu}{\sigma^2} \ \frac{-1}{2\sigma^2}]^T \rangle - \left(\frac{\mu^2}{\sigma^2} + \log \sigma\right) - \frac{1}{2} \log 2\pi\right\}$$

Minimal statistics $\mathbf{f}(x) = [x \ x^2]^T$, natural parameters $[\theta_1 \ \theta_2]^T = [\frac{\mu}{\sigma^2} \ \frac{-1}{2\sigma^2}]^T$,

cumulant function $F(\theta_1, \theta_2) = -\frac{\theta_1^2}{4\theta_2} - \frac{1}{2} \log(-2\theta_2)$, and $k(\mathbf{f}(x)) = \frac{1}{2} \log 2\pi$.

Duality [B+'05]

Bregman functions \leftrightarrow exponential families.

(Bijection primordial for designing tailored clustering divergences [B+05])

Kullback-Leibler divergence & Bregman divergences

Kullback-Leiber divergence $\text{KL}(p||q)$ of two probability distributions p and q : $\text{KL}(p||q) = \int_x p(x) \log \frac{p(x)}{q(x)} dx$.

Kullback-Leibler: Bregman divergence for the cumulant function

$\text{KL}(\theta_p||\theta_q) = D_F(\theta_q||\theta_p) = F(\theta_q) - F(\theta_p) - \langle (\theta_q - \theta_p), \theta_p[\mathbf{f}] \rangle$,
with $\theta_p[\mathbf{f}] = E_{p(x|\theta)}[\mathbf{f}(X)] = \boldsymbol{\mu}$ denoting *expectation parameters*.

$$\theta_p[\mathbf{f}] = [\int_x \mathbf{f}(x) \exp\{(\theta_p, \mathbf{f}(x)) - F(\theta_p) - k(\mathbf{f}(x))\} dx]$$

(Beware of argument swapping.)

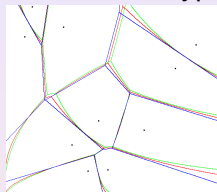
Kullback-Leibler and Legendre duality

$$\text{KL}(\theta_p||\theta_q) = D_F(\theta_q||\theta_p) = D_{F^*}(\boldsymbol{\mu}_p||\boldsymbol{\mu}_q).$$

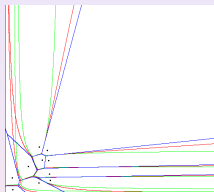
Bregman Voronoi diagrams: Bisectors

Two types of Voronoi diagrams defined by bisectors:

- First-type $H_{\mathbf{p}\mathbf{q}}$ $H(\mathbf{p}, \mathbf{q}) = \{\mathbf{x} \in \mathcal{X} \mid D_F(\mathbf{x} \parallel \mathbf{p}) = D_F(\mathbf{x} \parallel \mathbf{q})\}$ (linear)
- Second-type $H'_{\mathbf{p}\mathbf{q}}$: $H'(\mathbf{p}, \mathbf{q}) = \{\mathbf{x} \in \mathcal{X} \mid D_F(\mathbf{p} \parallel \mathbf{x}) = D_F(\mathbf{q} \parallel \mathbf{x})\}$ (curved)



(Kullback-Leibler)



(Itakura-Saito)

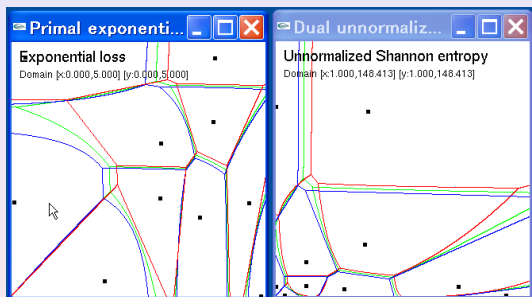
Affine/Curved Voronoi diagrams & Dualities

- Hyperplane $H(\mathbf{p}, \mathbf{q})$: $\langle \mathbf{x}, \mathbf{p}' - \mathbf{q}' \rangle + F(\mathbf{p}) - \langle \mathbf{p}, \mathbf{p}' \rangle - F(\mathbf{q}) + \langle \mathbf{q}, \mathbf{q}' \rangle = 0$.
- Hypersurface $H'(\mathbf{p}, \mathbf{q})$: $\langle \mathbf{x}', \mathbf{q} - \mathbf{p} \rangle + F(\mathbf{p}) - F(\mathbf{q}) = 0$.
Curved in \mathbf{x} but linear in \mathbf{x}' .

(Dual of first-type bisector for gradient point set \mathcal{S}' and D_{F^*} ; \Rightarrow image by ∇_F is a hyperplane in \mathcal{X}' .)

Duality: $\text{vor}_F(\mathcal{S}) \stackrel{\text{dual}}{\equiv} \text{vor}'_{F^*}(\mathcal{S}')$ and $\text{vor}'_F(\mathcal{S}) \stackrel{\text{dual}}{\equiv} \text{vor}_{F^*}(\mathcal{S}')$.

Bregman Voronoi diagrams: Videos



(Rasterized real-time on GPU, or computed exactly using the `qhull` package.)

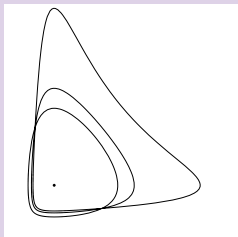
Visit <http://www.csl.sony.co.jp/person/nielsen/BregmanVoronoi>

Symmetrized Bregman divergences

Symmetrized Bregman divergence is a Bregman divergence

$$S_F(\mathbf{p}, \mathbf{q}) = \frac{1}{2}(D_F(\mathbf{p}||\mathbf{q}) + D_F(\mathbf{q}||\mathbf{p})) = \frac{1}{2}\langle \mathbf{p} - \mathbf{q}, \mathbf{p}' - \mathbf{q}' \rangle.$$

$$\begin{aligned} S_F(\mathbf{p}, \mathbf{q}) &= \frac{1}{2} (D_F(\mathbf{p}||\mathbf{q}) + D_F(\mathbf{q}||\mathbf{p})) \\ &= \frac{1}{2} (D_F(\mathbf{p}||\mathbf{q}) + D_{F^*}(\mathbf{p}'||\mathbf{q}')) \\ &= D_{\hat{F}}(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \end{aligned}$$



where $\hat{\mathbf{p}} = (\mathbf{p}, \mathbf{p}')$ and $\hat{F}(\hat{\mathbf{p}}) = \frac{1}{2} (F(\mathbf{p}) + F^*(\mathbf{p}'))$

Symmetrized bisector & Symmetrized Voronoi diagram

$$H_{S_F}(\mathbf{p}, \mathbf{q}) = \text{proj}_{\mathcal{X}} \left(H_{D_{\hat{F}}}(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \cap \mathcal{M} \right) \text{ with } \mathcal{M} = \{(\mathbf{x}, \mathbf{x}')\} \subset \mathbb{R}^{2d}$$

(Double space dimension: from d -variate F to $2d$ -variate \hat{F} .)



Space of Bregman spheres

Bregman spheres

$$\sigma(\mathbf{c}, r) = \{\mathbf{x} \in \mathcal{X} \mid D_F(\mathbf{x} \parallel \mathbf{c}) = r\}$$

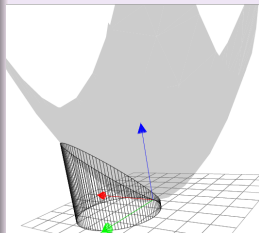
Lemma

The lifted image $\hat{\sigma}$ onto \mathcal{F} of a Bregman sphere σ is contained in a hyperplane H_σ .

$$(H_\sigma : z = \langle \mathbf{x} - \mathbf{c}, \mathbf{c}' \rangle + F(\mathbf{c}) + r)$$

Conversely, the intersection of any hyperplane H with \mathcal{F} projects vertically onto a Bregman sphere.

$$(H : z = \langle \mathbf{x}, \mathbf{a} \rangle + b \longrightarrow \sigma = (\nabla_F^{-1}(\mathbf{a}), \langle \nabla_F^{-1}(\mathbf{a}), \mathbf{a} \rangle - F(\nabla_F^{-1}(\mathbf{a})) + b))$$



(eg, usual paraboloid of revolution \mathcal{F})

Algorithmics of Bregman spheres/balls

Union/intersection of Bregman balls

The union/intersection of n Bregman balls of \mathcal{X} has complexity $\Theta(n^{\lfloor \frac{d+1}{2} \rfloor})$ and can be computed in time $\Theta(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor})$

- Boundary of $\bigcap_i \sigma_i$: Proj_\perp of $\mathcal{F} \cap \left(\bigcap_{i=1}^n H_{\sigma_i}^\uparrow \right)$.
- Boundary of $\bigcup_i \sigma_i$: Proj_\perp of complement of $\mathcal{F} \cap \left(\bigcap_{i=1}^n H_{\sigma_i}^\uparrow \right)$.

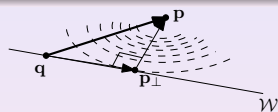
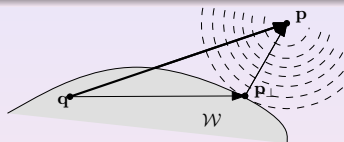
Generalize the (Euclidean) space of spheres to Bregman spaces of spheres: radical axes, pencils of spheres, etc.

(Important for manifold reconstruction since every solid is a union of balls; ie., medial axis — power crust)

Generalized Pythagoras theorem

Fact (Three-point)

$$D_F(\mathbf{p}||\mathbf{q}) + D_F(\mathbf{q}||\mathbf{r}) = D_F(\mathbf{p}||\mathbf{r}) + \langle \mathbf{p} - \mathbf{q}, \mathbf{r}' - \mathbf{q}' \rangle.$$



Fact (Bregman projection)

\mathbf{p}_\perp Bregman projection of \mathbf{p} onto convex subset $\mathcal{W} \subseteq \mathcal{X}$:

$$\mathbf{p}_\perp = \operatorname{argmin}_{\mathbf{w} \in \mathcal{W}} D_F(\mathbf{w}||\mathbf{p}).$$

Projection unique satisfying the Pythagoras theorem:

$$\langle \mathbf{p} - \mathbf{q}, \mathbf{p}'_\perp - \mathbf{q}' \rangle = 0.$$

Dual orthogonality of bisectors with geodesics

$$l(\mathbf{p}, \mathbf{q}) = \{\mathbf{x} : \mathbf{x} = \lambda\mathbf{p} + (1 - \lambda)\mathbf{q}\}$$
$$c(\mathbf{p}, \mathbf{q}) = \{\mathbf{x} : \mathbf{x}' = \lambda\mathbf{p}' + (1 - \lambda)\mathbf{q}'\}$$

Straight line segment $[p\mathbf{q}]$
Geodesic (\mathbf{p}, \mathbf{q})

Orthogonality (Projection)

X is Bregman orthogonal to Y if

$$\forall \mathbf{x} \in X, \forall \mathbf{y} \in Y, \exists \mathbf{t} \in X \cap Y$$

$$D_F(\mathbf{x}||\mathbf{t}) + D_F(\mathbf{t}||\mathbf{y}) = D_F(\mathbf{x}||\mathbf{y}) \Leftrightarrow \langle \mathbf{x} - \mathbf{t}, \mathbf{y}' - \mathbf{t}' \rangle = 0$$

Lemma

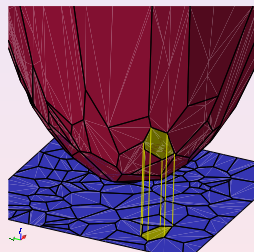
$c(\mathbf{p}, \mathbf{q})$ is Bregman orthogonal to $H_{\mathbf{p}\mathbf{q}}$

$l(\mathbf{p}, \mathbf{q})$ is Bregman orthogonal to $H'_{\mathbf{p}\mathbf{q}}$

Bregman Voronoi diagrams from polytopes

Theorem

The first-type Bregman Voronoi diagram $\text{vor}_F(\mathcal{S})$ is obtained by projecting by Proj_\perp the faces of the $(d+1)$ -dimensional polytope $\mathcal{H} = \cap_i H_{\mathbf{p}_i}^\uparrow$ of \mathcal{X}^+ onto \mathcal{X} .



(← Minimization diagrams)

The Bregman Voronoi diagrams of a set of n d -dimensional points have complexity $\Theta(n^{\lfloor \frac{d+1}{2} \rfloor})$ and can be computed in optimal time $\Theta(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor})$.

Bregman Voronoi diagrams from power diagrams

Affine Voronoi diagrams

The first-type Bregman Voronoi diagram of n sites of \mathcal{X} is identical to the power diagram of n Euclidean hyperspheres centered at $\nabla_F(\mathcal{S}) = \{\mathbf{p}' \mid \mathbf{p} \in \mathcal{S}\}$

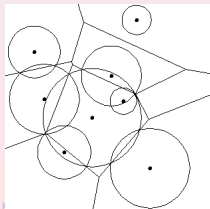
$$D_F(\mathbf{x}, \mathbf{p}_i) \leq D_F(\mathbf{x}, \mathbf{p}_j)$$

$$\iff -F(\mathbf{p}_i) - \langle \mathbf{x} - \mathbf{p}_i, \mathbf{p}'_i \rangle \leq -F(\mathbf{p}_j) - \langle \mathbf{x} - \mathbf{p}_j, \mathbf{p}'_j \rangle$$

$$\iff \langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{p}'_i \rangle - 2F(\mathbf{p}_i) + 2\langle \mathbf{p}_i, \mathbf{p}'_i \rangle \leq \langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{p}'_j \rangle - 2F(\mathbf{p}_j) + 2\langle \mathbf{p}_j, \mathbf{p}'_j \rangle$$

$$\iff \langle \mathbf{x} - \mathbf{p}'_i, \mathbf{x} - \mathbf{p}'_i \rangle - r_i^2 \leq \langle \mathbf{x} - \mathbf{p}'_j, \mathbf{x} - \mathbf{p}'_j \rangle - r_j^2$$

$$\mathbf{p}_i \rightarrow \sigma_i = \text{Ball}(\mathbf{p}'_i, r_i) \text{ with} \\ r_i^2 = \langle \mathbf{p}'_i, \mathbf{p}'_i \rangle + 2(F(\mathbf{p}_i) - \langle \mathbf{p}_i, \mathbf{p}'_i \rangle)$$



Straight triangulations from polytopes

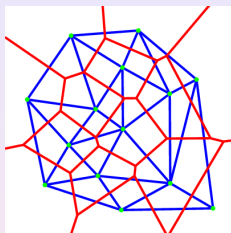
Several ways to define Bregman triangulations

Definition (Straight triangulation)

$\hat{\mathcal{S}}$: lifted image of \mathcal{S}

\mathcal{T} : lower convex hull of $\hat{\mathcal{S}}$

The vertical projection of \mathcal{T} is called the straight Bregman triangulation of \mathcal{S}

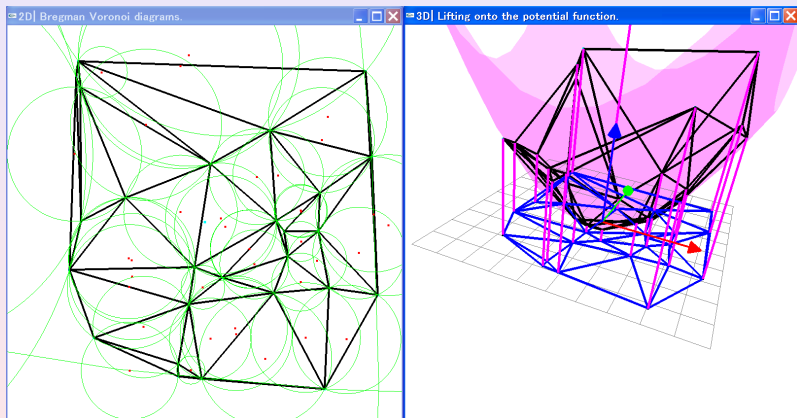


Properties

- Characteristic property : The Bregman sphere circumscribing any simplex of $BT(\mathcal{S})$ is empty
- Optimality : $BT(\mathcal{S}) = \min_{T \in \mathcal{T}(\mathcal{S})} \max_{\tau \in T} r(\tau)$
($r(\tau)$ = radius of the smallest Bregman ball containing τ)

[Rajan]

Bregman Voronoi diagram & triangulation from polytopes

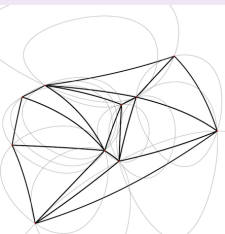


(Implemented using OpenGL®.)



Bregman triangulations: Geodesic triangulations

The straight triangulation is not necessarily the dual of the Bregman Voronoi diagram. The dual triangulation is the *geodesic triangulation* (bisector/geodesic Bregman orthogonality).



Geodesic triangulation

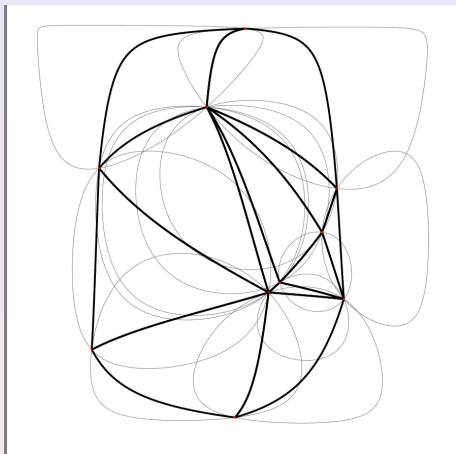
- Image of triangulation by ∇_F^{-1} is a curved triangulation
- Edges: geodesic arcs joining two sites

Symmetric divergences

For symmetric divergences, straight and geodesic triangulations are combinatorially equivalent (polarity).

(For $D_F = L_2^2$, straight and geodesic triangulations are the same ordinary Delaunay triangulations.)

Geodesic triangulation: Hellinger-type



(Hellinger distance $D^{(0)}(\rho||q) = 2 \int (\sqrt{\rho(x)} - \sqrt{q(x)})dx$, a particular case of f -divergence)

($D^{(1)}$ yields Kullback-Leibler)

Riemannian geometry

Bregman geometry can be tackled from the viewpoint of Riemannian geometry:

- Riemannian geometry & Information geometry [AN'00],
(mostly Fischer metrics for statistical manifolds)
- Calculus on manifolds: Parallel transports of vector fields and connections,
- geodesic Voronoi diagram $\int_a^b \sqrt{\mathbf{g}_{ij} \gamma^{i'} \gamma^{j'}} ds$ (metric \mathbf{g}_{ij})
- Canonical divergences & second- or third-order metric approximations

$$(\mathbf{g}^{(D_F)} = \mathbf{g}^{(D_{F^*})} \text{ with } \mathbf{g}_{ij}^{(D_F)} = -D_F(\partial_i || \partial_j) = D_F(\partial_i \partial_j || \cdot) = D_F(\cdot || \partial_i \partial_j))$$

Bregman geometry: A special case of Riemannian geometry

- Dual affine coordinate systems (\mathbf{x} and \mathbf{x}') of *non-metric* connections ∇ and ∇^* (D_F and D_{F^*} are non-conformal representations)
- Torsion-free & zero-curvature space (dually flat space).

In our paper, we further consider

- Weighted Bregman diagrams:
 $WD_F(\mathbf{p}_i || \mathbf{p}_j) = D_F(\mathbf{p}_i || \mathbf{p}_j) + w_i - w_j$
(incl. k -order Bregman diagrams)
- k -jet Bregman divergences (tails of Taylor expansions)
- Centroidal Voronoi diagrams (centroid & Bregman information).
- Applications to ϵ -sampling and minmax quantization
- Applications to machine learning

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Thank You!

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