Further results on the hyperbolic Voronoi diagrams

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***Abstract*—In Euclidean geometry, it is well-known that the k-order Voronoi diagram in $\mathbb{R}^d$ can be computed from the vertical projection of the k-level of an arrangement of hyperplanes tangent to a convex potential function in $\mathbb{R}^{d+1}$: the paraboloid. Similarly, we report for the Klein ball model of hyperbolic geometry such a *concave* potential function: the northern hemisphere. Furthermore, we also show how to build the hyperbolic k-order diagrams as equivalent clipped power diagrams in $\mathbb{R}^d$. We investigate the hyperbolic Voronoi diagram in the hyperboloid model and show how it reduces to a Klein-type model using central projections.**

***Keywords*-Voronoi diagram; hyperbolic geometry; clipping.**

I. Introduction

Hyperbolic geometry is a consistent geometry where the Euclidean Playfair's parallel postulate is discarded and replaced by the existence of many lines U not intersecting another given line L and passing through a given point $P \notin L$ (the U's are said *ultra-parallel*[1] to L). Hyperbolic geometry can be studied using various models [1]: Poincaré disk or upper plane conformal models, Klein non-conformal model disk model, hyperboloid conformal model, etc. From the viewpoint of computational geometry, we prefer to use Klein model where lines/bisectors are Euclidean straight [2] and then convert the output to the desired model for visualization or navigation purposes [1]. We report further novel results for constructing hyperbolic Voronoi diagrams (HVDs) in Klein model [2] and present yet another approach to get Klein-type affine bisectors/diagrams from the hyperboloid[2] model.

II. HVDs from lower envelopes

The *Voronoi diagram* of a set $\mathcal{P} = \{p_1, ..., p_n\}$ of n points in $\mathbb{R}^d$ w.r.t. $D(\cdot,\cdot)$ can be computed equivalently as the *minimization diagram* of n functions by observing that $D(x,p_i) \leq D(x,p_j) \Leftrightarrow F_i(x) \leq F_j(x)$ where $F_l(x) = D(x,p_l)$, $l \in \{1,...,n\}$. Thus the *combinatorial structures* are congruent: $\mathrm{Vor}_D(\mathcal{P}) \cong \min_{l\in\{1,...,n\}} F_l(x)$. Furthermore, this minimization diagram amounts to compute the *lower envelope* of n graph functions in $\mathbb{R}^{d+1}$: $\mathcal{F}_l : \{(x, y = F_l(x)) \ : \ x \in \mathbb{R}^d\}$.

[1] *Parallel* lines intersect at infinity in hyperbolic geometry.
[2] Hyperbolic geometry stems from the hyperboloid model.

Let $\langle x,p\rangle = x^\top p = \sum_{i=1}^d x^{(i)} p^{(i)}$ denotes the Euclidean inner product. In the Klein model [2], the distance between two points x and p in the open unit ball domain $\mathbb{B}_d = \{x \in \mathbb{R}^d : \langle x,x\rangle < 1\}$ is $D^K(x,p) = \mathrm{arccosh}\frac{1-\langle x,p\rangle}{\sqrt{1-\langle x,x\rangle}\sqrt{1-\langle p,p\rangle}}$ where $\mathrm{arccosh}(x) = \log(x+\sqrt{x^2-1})$ for $x \geq 1$ is a monotonically increasing function. Since the Voronoi diagram does not change by composing the distance with a monotonous function, we consider the equivalent Klein distance $d^K(x,p) = \frac{1-\langle x,p\rangle}{\sqrt{1-\langle x,x\rangle}\sqrt{1-\langle p,p\rangle}}$. To each point $p_i \in \mathcal{P}$ corresponds a function $F_i(x) = d^K(x,p_i)$. Since the denominator $\sqrt{1-\langle x,x\rangle}$ is common to all functions, the minimization diagram is equivalent to the minimization diagram of $F_i'(x) = \frac{1-\langle x,p_i\rangle}{\sqrt{1-\langle p_i,p_i\rangle}}$. The graph $\mathcal{F}_i' = \{(x, y = F_i(x)) \ : \ x \in \mathbb{B}_d\}$ are *hyperplanes* in $\mathbb{R}^{d+1}$ defined on $\mathbb{B}_d$, and the lower envelope can thus be computed from the intersection of n halfspaces $H_i^- : y \leq \frac{1-\langle x,p_i\rangle}{\sqrt{1-\langle p_i,p_i\rangle}}$, yielding the Voronoi unbounded polytope in $\mathbb{R}^{d+1}$.

Theorem 1: The HVD of n points can be computed in the Klein model as the intersection of n half-spaces in $\mathbb{R}^{d+1}$ and by projecting vertically ($\downarrow H_0 : y = 0$) the polytope on $\mathbb{R}^d$, and clipping it with the unit ball domain: $\mathrm{Vor}_{d^K}(\mathcal{P}) = ((\cap_{i=1}^n H_i^-) \downarrow H_0) \cap \mathbb{B}_d$.

III. Lifting sites to a potential function

In Euclidean (and more generally Bregman geometry), the Voronoi polytope is built by lifting points to tangent hyperplanes to a *potential function* $y = F(x)$ at site locations. This is the paraboloid lifting transformation: $y = F(x) = \langle x,x\rangle$ ($y = F(x)$ for a convex Bregman generator F).

Theorem 2: In the Klein ball model, the *potential function* for lifting generators to hyperplanes is the *concave* function $y = F(x) = \sqrt{1-\langle x,x\rangle}$ restricted to $\mathbb{B}_d$.

Proof: Let us identify the hyperplane equation $H(p)$: $y = \frac{1-\langle p,x\rangle}{\sqrt{1-\langle p,p\rangle}}$ with the hyperplane tangent at p to a potential function $y = F(x)$: $\langle\nabla F(p), x-p\rangle + F(p) = \langle x, \nabla F(p)\rangle + F(p) - \langle p, \nabla F(p)\rangle$. We have $\nabla F(p) = -\frac{p}{\sqrt{1-\langle p,p\rangle}}$ and the remaining term (independent of x) is $\frac{1}{\sqrt{1-\langle p,p\rangle}}$. The anti-derivative of $\nabla F(x) = -\frac{x}{\sqrt{1-\langle x,x\rangle}}$ is $\sqrt{1-\langle x,x\rangle} + c$, and

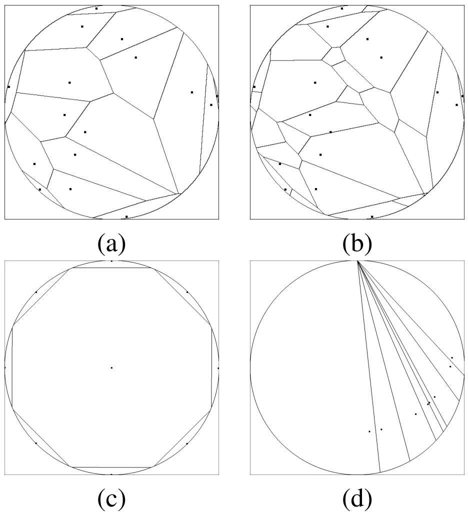

Figure 1. HVD for $k = 1$ (a) and $k = 2$ (b). HVD with all unbounded cells (c), and pencil of parallel bisectors intersecting at $\partial\mathbb{B}_d$ (d).

the constant c solves to zero. This is the equation $y^2 + \langle x, x\rangle = 1$ of the northern hemisphere for $y \geq 0$. Observe that the hyperplanes tend to become vertical as we near the boundary domain $\partial\mathbb{B}_d$, and are vertical at the boundary.

IV. k-ORDER HYPERBOLIC VORONOI DIAGRAMS

Since the Klein bisector is affine, the k-order HVD is affine. We present two construction methods.

A. *k-HVDs from levels of an arrangement of hyperplanes*

This is a straightforward generalization of the Euclidean procedure using the $\sqrt{1-\langle x,x\rangle}$ potential function. The k-order HVD is a *cell complex* that can be built by projecting to $\mathbb{R}^d$ all the $(d+1)$-dimensional cells at k-level of the arrangement of the site hyperplanes $\mathcal{H} : \{H_1, ..., H_n\}$ of $\mathbb{R}^{d+1}$ and clipping the structure to $\mathbb{B}_d$. Figure 1 displays some k-order diagrams and illustrates some degenerate cases.

B. *k-HVDs from power diagrams*

Consider all subsets of size k, $\mathcal{P}_k = \binom{\mathcal{P}}{k} = \{\mathcal{K}_1, ..., \mathcal{K}_N\}$ with $N = \binom{n}{k}$. Those *subset generators* partition the space into *non-empty k-order Voronoi cells*:

$$\mathrm{Vor}_k(\mathcal{K}_i) = \{x : \forall q \in \mathcal{K}_i, \forall r \in \mathcal{P}\backslash\mathcal{K}_i,\ D(x,q) \leq D(x,r)\}.$$

Observe that $x \in \mathrm{Vor}_k(\mathcal{K}_i)$ iff $\sum_{p\in\mathcal{K}_i} D(x,p) \leq \sum_{p'\in\mathcal{K}_j} D(x,p')$. In Klein model with $D = d^K$, we define the function $\sigma_{\mathcal{K}_i}(x) = \sum_{x\in\mathcal{K}_i} \frac{1-\langle x,p_i\rangle}{\sqrt{1-\langle p_i,p_i\rangle}}$, and $x \in \mathrm{Vor}_k(\mathcal{K}_i) \Leftrightarrow h_{\mathcal{K}_i}(x) \leq h_{\mathcal{K}_j}(x)\ \forall j \neq i$. By identifying those hyperplane equations with the generic power diagram hyperplane $h(x) : y = -2\langle x,c\rangle - w + \langle c,c\rangle$ for a ball centered at c and radius $r^2 = w$ (r may be imaginary when $w < 0$), we transform each k-subset $\mathcal{K}_i$ in Klein model into a weighted point (or ball) $\mathrm{ball}(c_i, w_i)$: $c_i = \sum_{p\in\mathcal{K}_i} \frac{p}{2\sqrt{1-\langle p,p\rangle}}$ and $w_i = \langle c_i, c_i\rangle - \sum_{p\in\mathcal{K}_i} \frac{1}{\sqrt{1-\langle p,p\rangle}}$. This method is only practical if when we consider all subsets $\mathcal{K}_i$ that yields non-empty cells, otherwise we have $N = \binom{n}{k}$ too many balls to be tractable!

V. HVDS FROM THE HYPERBOLOID MODEL

Consider the symmetric bilinear form $L = \mathrm{diag}(-1, 1, ..., 1)$ in Minkowski space $\mathbb{R}^{1,d}$: $\langle p,q\rangle_L = p^\top L q = -p^{(0)}q^{(0)} + \sum_{i=1}^d p^{(i)}q^{(i)}$. The hyperboloid model is defined on the upper sheet domain $\mathbb{L}^+ = \{\langle x,x\rangle_\mathrm{L} = -1,\ x_0 > 0\}$ (interpreted as a sphere $\langle x,x\rangle_\mathrm{L} = R^2$ of imaginary radius $R = i$). For $x \in \mathbb{R}^d$, we denote x^L its point obtained by vertically rising $(\cdot, x)$ on $\mathbb{L}^+$: $x^L = (\sqrt{1+\langle x,x\rangle}, x)$, called Weierstrass coordinates. The hyperbolic distance is expressed by $D^L(p^L, q^L) = \mathrm{arccosh}(-\langle p^L, q^L\rangle_\mathrm{L})$ and is equivalent to $d^L(p^L, q^L) = -\langle p^L, q^L\rangle_\mathrm{L}$. For two points p^L and q^L on $\mathbb{L}^+$, the bisector equation is $\langle x^L, p^L - q^L\rangle_\mathrm{L} = 0$. The bisector is an hyperbola of equation $\left(\sqrt{1+\langle p,p\rangle} - \sqrt{1+\langle q,q\rangle}\right)\sqrt{1+\langle x,x\rangle} + \langle q-p, x\rangle = 0, x \in \mathbb{R}^d$ $(*)$. This hyperbola bisector is contained in a hyperplane $H(p,q)$ of $\mathbb{R}^{d+1}$ passing through the origin O: $H(p,q) : (\sqrt{1+\langle p,p\rangle} - \sqrt{1+\langle q,q\rangle})x_0 + \langle q-p, x\rangle = 0$. The Klein disk model is obtained from $\mathbb{L}^+$ by a central projection π from the origin to the hyperplane $H_1 : x_0 = 1$: $\pi\begin{bmatrix} x_0 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ x' = \frac{x}{x_0} = \frac{x}{\sqrt{1+\langle x,x\rangle}} \end{bmatrix}$. The disk center touches the apex of $\mathbb{L}^+$. Let $a_{p,q} = \sqrt{1+\langle p,p\rangle} - \sqrt{1+\langle q,q\rangle}$. Multiplying $(*)$ by $\frac{1}{\sqrt{1+\langle x,x\rangle}}$, we have the bisector written as $\langle q-p, x'\rangle + a_{p,q} = 0$, an affine bisector in x'.

Now consider $\pi_{c,l}$ the *generic* central projection of $\mathbb{L}^+$ from $C = (c, 0)$ to the hyperplane $H_l : x_0 = l$ so that $\pi = \pi_{0,1}$. We have $\pi_c\begin{bmatrix} \sqrt{1+\langle x,x\rangle} \\ x \end{bmatrix} = \begin{bmatrix} l \\ x_{c,l} = \frac{l-c}{\sqrt{1+\langle x,x\rangle}-c}x \end{bmatrix}, c \neq 1$. Choosing $c = 0$ and $0 < l \leq 1$ yields the same construction procedure but the clipping of the equivalent power diagram [2] need to be done on a disk of size l since $\|x_{c,l}\| = \|\frac{l}{\sqrt{1+\langle x,x\rangle}}x\| \leq l$, $\forall x \in \mathbb{R}^d$. Note that clipping may destroy many bounded cells of the affine diagram. Thus an open question is to report an optimal output-sensitive construction of the k-order HVDs.

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